Probability problems solved and calculated with CASIO calculators (and also proved using mathematics)

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CASIO

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Thanks to Villő Csiszár for the great problems.

1. Using the sum function

Problem 1.1.

I roll a red and a white dice alternately and I started with the red.

I repeat the rolling process until it will be even with the red die or a number divisible by three on the white one. Let us denote X the number of rolls.

Give the distribution and the mean of X!

Solution of Problem 1.1.

Let us write a table with the probabilities.

It is trivial that the probability of an even number is 1/2, since P(2 or 4 or 6) = 1/2. Similar we can state that P(3 or 6) = 1/3.

For example let us see the case when we finish the process at the 5th roll. It means that the probabilities of the rolls are 1/2, 2/3, 1/2, 2/3 and finally 1/2. The events are independent so the probability is the product of the numbers 1/2 · 2/3 · 1/2 · 2/3 · 1/2 = 1/18.

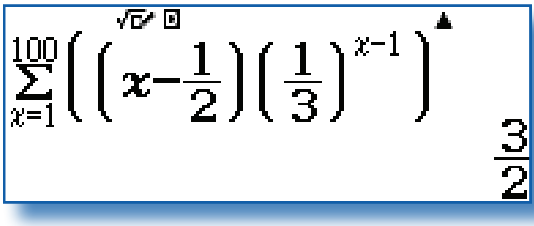
	Number of rolls	Probability
R_1	1	1/2
W_1	2	1/2 · 1/3 = 1/6
R_2	3	1/2 · 2/3 · 1/2 = 1/6
W_2	4	1/2 · 2/3 · 1/2 · 1/3 = 1/18
R_3	5	1/2 · 2/3 · 1/2 · 2/3 · 1/2 = 1/18
W_3	6	1/2 · 2/3 · 1/2 · 2/3 · 1/2 · 1/3 = 1/54
…	…	…
R_i	$2i-1$	$(1/3)^{i-1} \cdot 1/2$
W_i	$2i$	$(1/3)^{i} \cdot 1/2$
…	…	…

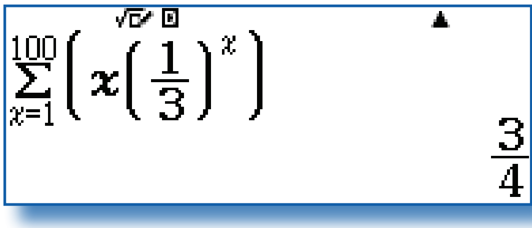
The expected value (the mean) is $E=\sum_{i=1}^{\infty}(2i-1)\cdot\left(\frac{1}{3}\right)^{i-1}\cdot\frac{1}{2}+\sum_{i=1}^{\infty}2i\cdot\left(\frac{1}{3}\right)^{i}\cdot\frac{1}{2}$

There are a lot of different ways to get the sum. The quickest way is using the sum function of the calculator. We use 100 instead of infinity in the sum function.

1. Using the sum function

The two sums are





So the mean is $\frac{3}{2}+\frac{3}{4}=\frac{9}{4}=2.25$.

Extension:

We could get the same result after some basic calculation. We recall the sum of a geometric series

$$1+p+p^2+p^3+\ldots=\frac{1}{1-p}\quad \text{if } |p|<1$$

and the mean of a geometric (Pascal) distribution.

We are say X is a geometric (Pascal) distribution with parameter p if

$$P(X=i+1)=(1-p)^i\cdot p \quad \text{where } i=0,\ldots,\infty$$

The expected value of X is $E(X)=\frac{1}{p}$.

In this situation we reorder the sums. It is just simple algebra and we can get

$$E=\sum_{i=1}^{\infty}(2i-1)\cdot\left(\frac{1}{3}\right)^{i-1}\cdot\frac{1}{2}+\sum_{i=1}^{\infty}2i\cdot\left(\frac{1}{3}\right)^{i}\cdot\frac{1}{2}=$$

$$=\sum_{i=1}^{\infty}(2i-1)\cdot\left(\frac{1}{3}\right)^{i-1}\cdot\frac{1}{2}+2i\cdot\left(\frac{1}{3}\right)^{i-1}\cdot\frac{1}{6}=$$

$$=\sum_{i=1}^{\infty}\left(\frac{4}{3}i-\frac{1}{2}\right)\cdot\left(\frac{1}{3}\right)^{i-1}=\sum_{i=1}^{\infty}\frac{4}{3}i\cdot\left(\frac{1}{3}\right)^{i-1}-\sum_{i=1}^{\infty}\frac{1}{2}\cdot\left(\frac{1}{3}\right)^{i-1}=$$

$$=2\cdot\underbrace{\sum_{i=1}^{\infty}i\frac{2}{3}\cdot\left(\frac{1}{3}\right)^{i-1}}_{*}-\underbrace{\frac{1}{2}\cdot\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i-1}}_{**}=2\cdot\frac{3}{2}-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{3}}=3-\frac{3}{4}=\frac{9}{4}$$

* is the double of the expected value of a geometric distribution with parameter $p = 2/3$.

** is the half of the sum of a geometric series.

We got the same result as earlier with the sum function of the calculator of course.

Problem 1.2.

There is a national lottery in Hungary. The gamer marks 5 numbers among 90 and the lottery draws 5 numbers out of the 90. After the draw the national lottery gives the five numbers in monotonic increasing order. Let us denote the five numbers by $x_1 < x_2 < x_3 < x_4 < x_5$.

a) The basic question is the mean of the middle number, $E(x_3)$.

b) The main question is the mean of the smallest number, $E(x_1)$?

Comment: We are not dealing now with the probability of a winning situation. It is an easy and solvable task for any student.

Solution of Problem 1.2.

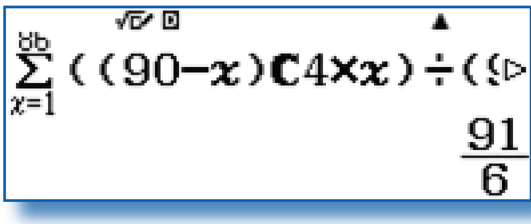
a) It is easy to show that $E(x_3) = 45.5$ because of the symmetry reason.

b) It is not as trivial at the first sight as problem a). Let us write the probability of the smallest number is i and the expected value of x_3:

$$P(i) = \frac{\binom{90-i}{4}}{\binom{90}{5}} \qquad E(X_1) = \frac{\sum_{i=1}^{86} i \cdot \binom{90-i}{4}}{\binom{90}{5}}$$

With calculator you can get $E(X_1) = \frac{91}{6} = 15\frac{1}{6}$.

(Be careful with the formula, you should use parenthesis on the appropriate places.)



Extension:

What shell we do without calculator? Let's see only the numerator of the fraction.

$$\sum_{i=1}^{86} i \cdot \binom{90-i}{4} = 1 \cdot \binom{89}{4} + 2 \cdot \binom{88}{4} + 3 \cdot \binom{87}{4} + \ldots + 84 \cdot \binom{6}{4} + 85 \cdot \binom{5}{4} + 86 \cdot \binom{4}{4}$$

1. Using the sum function

In the last part we can write $\binom{5}{5}$ instead of $\binom{4}{4}$, and we write 86 = 85 + 1 in the same step. After it we can use the rule of the Pascal-triangle, $\binom{5}{4}+\binom{5}{5}=\binom{6}{5}$. If we repeat the same process we get

$$\sum_{i=1}^{86} i\cdot\binom{90-i}{4}=1\cdot\binom{89}{4}+2\cdot\binom{88}{4}+3\cdot\binom{87}{4}+\ldots+84\cdot\binom{6}{4}+\underbrace{85\cdot\binom{5}{4}+85\cdot\binom{5}{5}}_{85\cdot\binom{6}{5}}+\binom{5}{5}=$$

$$=1\cdot\binom{89}{4}+2\cdot\binom{88}{4}+3\cdot\binom{87}{4}+\ldots+84\cdot\binom{6}{4}+84\cdot\binom{6}{5}+\binom{6}{5}+\binom{5}{5}=$$

$$=\binom{90}{5}+\binom{89}{5}+\binom{88}{5}+\ldots+\binom{8}{5}+\binom{7}{5}+\binom{6}{5}+\binom{5}{5}=\binom{91}{6}$$

We used the so called sock (or boots) rule of Pascal's triangle. The expected value is $\frac{\binom{91}{6}}{\binom{90}{5}}=\frac{91}{6}=15\frac{1}{6}$ as it was calculated earlier with the Casio calculator.

Remark 1.: If we think it deeper then we can see 5 numbers in a symmetric situation between 1 and 90. The numbers are trying to distribute in the whole interval uniformly and the 5 numbers create six equal length intervals. Their places are $15\frac{1}{6}$; $30\frac{2}{6}$; $45\frac{3}{6}$; $60\frac{4}{6}$; $75\frac{5}{6}$.

Remark 1.2.

From 1957 there were 3170 lottery weeks till 07.12.2017 in Hungary and the empirical averages are: 15.03470032; 29,9340694; 45.66340694; 60.5659306; 75.48832808

CASIO

Problem 1.3.

We have the numbers on an octahedron from 1 to 8. We will roll with it till the number will be greater than 5. Give the average sum of the rolled numbers.

Solution of Problem 1.3.

It is trivial that the probability of every elementary event is 1/8. On the other hand it is obvious that we have no chance to write the probabilities of the sums. There are numerous different ways to get for example 16. Let us denote the random sum by X.

The solution is closer if we remember on the law of total expectation.

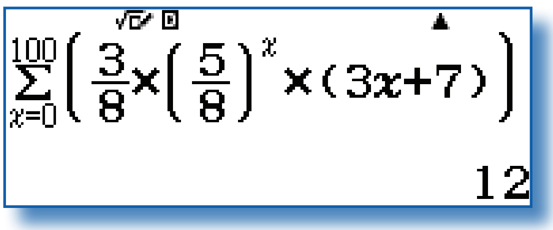
It states that if $\{A_j\}$ is a partition of the sample space and we know $E(X|A_j)$ then we can calculate $E(X)=\sum_j E(X\mid A_j)\cdot P(A_j)$ where j goes up to a finite number or infinity depends on the sample space.

Quite easy to calculate the probability that we can end it on the n-th step. These events are mutually exclusive and their sum is the whole space, so they will be A_j.

Number of rolls	Probability of A_j	Average if A_j
1	3/8	7
2	5/8 · 3/8	3 + 7
3	5/8 · 5/8 · 3/8	3 + 3 + 7
4	5/8 · 5/8 · 5/8 · 3/8	3 + 3 + 3 + 7
…	…	…
i	$(5/8)^{i-1}\cdot 3/8$	$3(i-1)+7$
$i+1$	$(5/8)^{i}\cdot 3/8$	$3i+7$
…	…	…

We get that $E(X)=\sum_{i=0}^{\infty}\frac{3}{8}\cdot\left(\frac{5}{8}\right)^{i}\cdot(3i+7)$.

Calculating with the Casio sum function we get:



Extension:

We did a similar algebraic transformation earlier and after some calculation we can get the same result.

$$E(X)=\sum_{i=0}^{\infty}\frac{3}{8}\cdot\left(\frac{5}{8}\right)^{i}\cdot(3i+7)=3\cdot\frac{5}{8}\cdot\underbrace{\sum_{i=1}^{\infty}\frac{3}{8}\cdot\left(\frac{5}{8}\right)^{i-1}\cdot i}_{*}+7\cdot\frac{3}{8}\cdot\underbrace{\sum_{i=0}^{\infty}\left(\frac{5}{8}\right)^{i}}_{**}=3\cdot\frac{5}{8}\cdot\frac{8}{3}+7\cdot\frac{3}{8}\cdot\frac{8}{3}=5+7=12$$

* This sum is the expected value of a geometric distribution with parameter 3/8.

** This is a standard geometric series.

2. The central limit theorem

It is known that if we repeat the same experiment and sum up the results then we can get a normal distribution. In a slightly appropriate way, the sum of some independent random variables gives a new random variable with normal distribution. If we normalize it then the result will be approximately a standard normal distribution.

Source (wikimedia)

Problem 2.1.

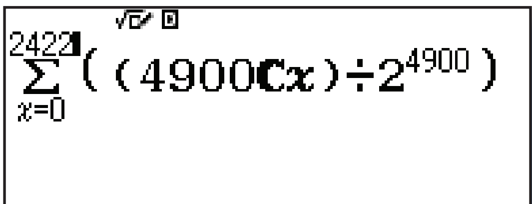
Give the estimation of the probability that you get at most 2422 head from 4900 tosses.

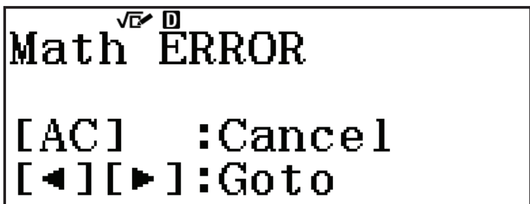
Solution of Problem 2.1.

It is trivial that the exact probability of this event is

$$\sum_{i=0}^{2422} \binom{4900}{i} \cdot \left(\frac{1}{2}\right)^{i} \cdot \left(\frac{1}{2}\right)^{4900-i} \approx 0.216$$

but if we put it in the Casio calculator we got an ERROR message.





2. The central limit theorem

It is normal, because $\binom{4900}{i}$ a huge number if i is big enough and the calculator can not handle these numbers.

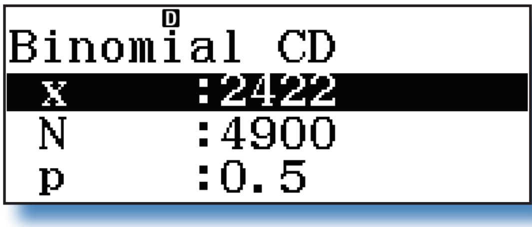
How can we get the result?

Let us choose the distribution menu under code 7 then go to the second page and select the binomial cumulative distribution and the parameters option. Finally

MENU 7 ▼ 1 2

We can give the parameters of the distribution and we get the answer. This is a proper answer.

$$\sum_{i=0}^{2422} \binom{4900}{i} \cdot \left(\frac{1}{2}\right)^{i} \cdot \left(\frac{1}{2}\right)^{4900-i} \approx 0.216$$



If we have no chance to use the binomial distribution than we can use the Central limit theorem.

Extension:

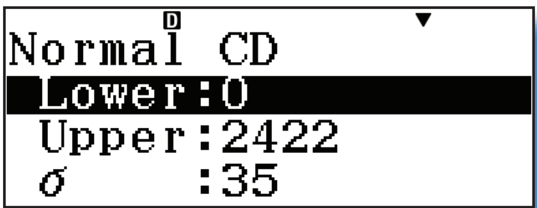
Let us denote the number of the heads from 4900 tosses by X. This random variable is binomial and it is the sum of 4900 pieces of Bin(1; 0.5) which is quite normal. If we get the standardized value of it then it will be an approximately standard normal distribution. There are many different possible ways to denote the expected value and the standard deviation. We use the Greek letter μ and σ respectively in the case of normal distribution, so we say there is an $N(\mu, \sigma^2)$ distribution. In this special case the mean of X is $\mu = E(X) = n \cdot p = 4900 \cdot 0.5 = 2450$ and the deviation is $\sigma = D(X) = \sqrt{npq} = \sqrt{4900 \cdot 0.5 \cdot 0.5} = 35$.

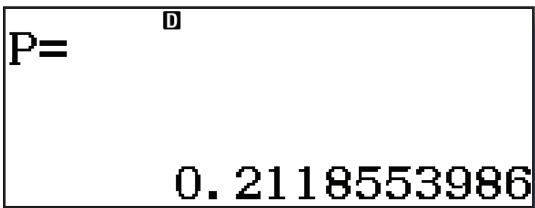
$$P(X < 2422) = P\left(\frac{X-\mu}{\sigma} < \frac{2422-2450}{35}\right) = P(X_{std} < -0.8) = \Phi(-0.8) = 1 - \Phi(0.8) \approx 0.212$$

The 0.212 result is quite good for the calculated one. The difference is only 0.004 is less than 2%.

MENU 7 2

You can get the answer with the Central limit theorem also from the CASIO calculator. Choose the cumulated normal distribution.





2. The central limit theorem

Problem 2.2.

There are 100 guests on the first day invited for welcome dinner at a conference. Everyone can choose between the meals A and B. We suppose that the two meals are equally preferred. How many portions should be prepare from the two meals that the probability that everyone can choose his/her preferred meal is at least 90%?

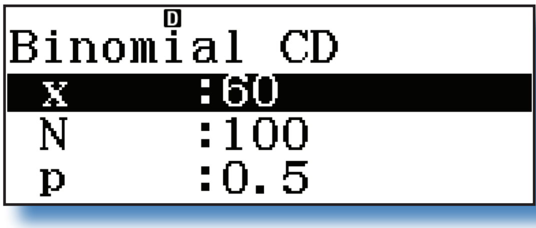
Solution of Problem 2.2.

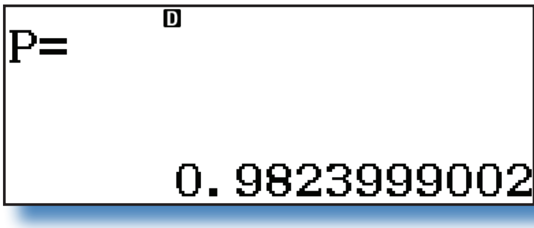
Let is denote by Z how many people choose the meal A. In this case Z has a binomial distribution where $n = 100$ and $p = 0.5$. We should calculate the number x, when the sum of the probability is greater then 0.9.

$$\sum_{i=0}^{x} \binom{100}{i} \cdot 0.5^{i} \cdot 0.5^{100-i} > 0.9$$

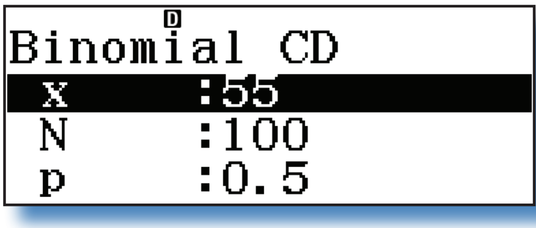
We can not calculate it directly by a Casio calculator, but after some good trial we get the exact value of k. Let us choose the menu 7 and the cumulative binomial distribution.

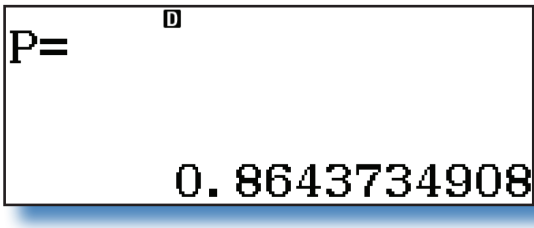
Let start the attempt by $x = 60$. The probability 0.98... is too much.





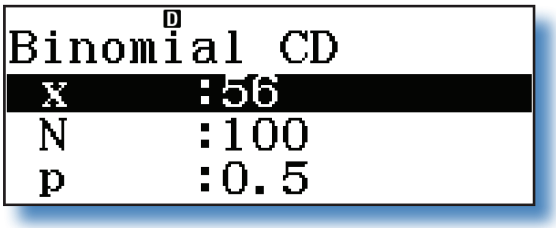
Our next try is $x = 55$, but we get that the probability is less than 0.9.

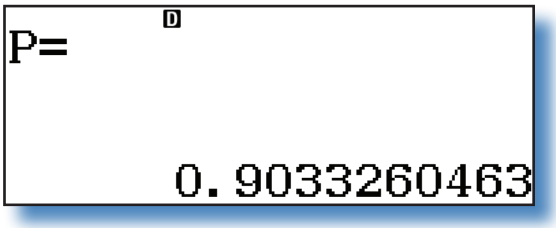




2. The central limit theorem

The try $x = 56$ will be perfect because it is the first number when the probability is greater than 0.9.





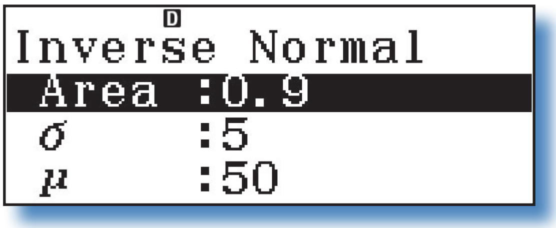
We can use the Central limit theorem instead of guessing, and it will give an estimation.

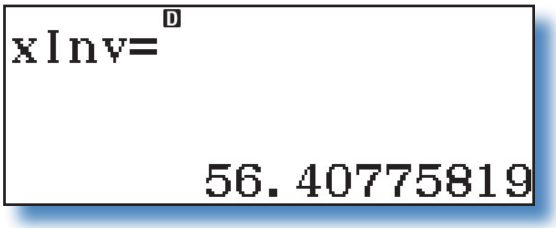
The mean and the deviation of the distribution are

$$E(Z) = np = 100 \cdot 0.5 = 50 \quad \text{and} \quad D(Z) = \sqrt{npq} = \sqrt{100 \cdot 0.5 \cdot 0.5} = \sqrt{25} = 5$$

and we are looking for the appropriate x, when $P(Z < x) > 0.9$. The calculator can show the value of x. Let us choose inverse normal option from the menu

We give the wished probability 0.9, the expected value (μ) and the deviation (σ).





The limit of x is 56.4, which is fairly good using the Central limit theory. It means that if we prepare 57 portions from both types of dishes, then everyone can choose his preferred meal at least with a 90% probability.

Extension:

We get the same result if we use the standardization and some basic calculation with the standardized data. In that case the expected value $\mu = 0$ and the standard deviation $\sigma = 1$.

$$P\left(\frac{Z-E}{D} < \frac{x-50}{5}\right) > 0.9$$

$$P\left(Z_{\text{std}} < \frac{x-50}{5}\right) > 0.9$$

$$\Phi\left(\frac{x-50}{5}\right) > 0.9$$

$$\frac{x-50}{5} > \Phi^{-1}(0.9) \approx 1.2816$$

$$x > 5 \cdot 1.2816 + 50$$

$$x > 56.408$$

3. Markov and Chebyshev inequalities

We recall the two famous inequalities.

Markov inequality
If X is any non-negative random variable, where $E(X)$ is existing then

$$P(X \geq a) \leq \frac{E(X)}{a}, \quad \text{for any } a > 0.$$

Chebyshev inequality
If X is any random variable, where $E(X)$ and $D(X)$ are existing then

$$P\left(\left|X - E(X)\right| \geq b\right) \leq \frac{D^2(X)}{b^2}, \text{ for any } b > 0.$$

Problem 3.1.

200 shots are dropped on a target. The probability of a hit is 0.4 in each independent shot. Let us denote X the number of hits. Estimate the probability $P(X \geq 92)$.

Solution of Problem 3.1.

The sum of the hits is binomial with $E(X) = np = 200 \cdot 0.4 = 80$ and $D^2(X) = np(1-p) = 200 \cdot 0.4 \cdot 0.6 = 48$.

a) Using the Markov inequality we get $P(X \geq 92) = P(X \geq 92) \leq \frac{80}{92} = \frac{20}{23} \approx 0.8696$. Everyone can feel that this estimation is really poor.

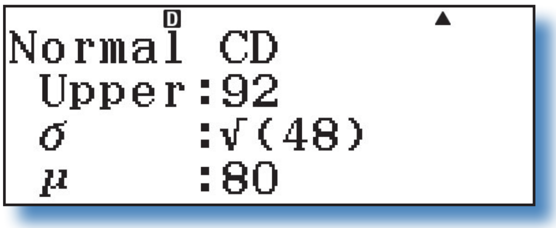
b) Let us use the Chebysev inequality:

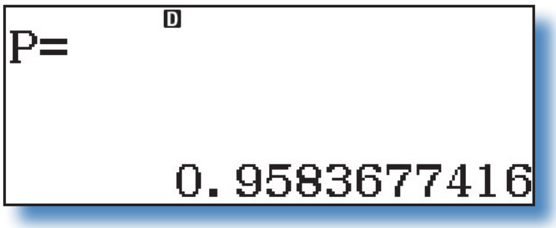
$$P(X \geq 92) = P(X - E \geq 92 - 80) = P(X - E \geq 12) = \frac{1}{2} P\left(\left|X - E\right| \geq 12\right) \leq \frac{1}{2} \cdot \frac{48}{12^2} = \frac{1}{6} \approx 0.1667$$

This estimation is much better, but still far from the reality.

c) If we use the Central limit theorem than we can get a fairly good estimation.
In the menu 7 choose the 2nd option (Normal CD) and let the parameter Lower: −1000.
Do not forget that $P(X \geq 92) = 1 - P(X < 92) = 1 - F(92)$ and the calculator gives $F(92)$.

3. Markov and Chebyshev inequalities

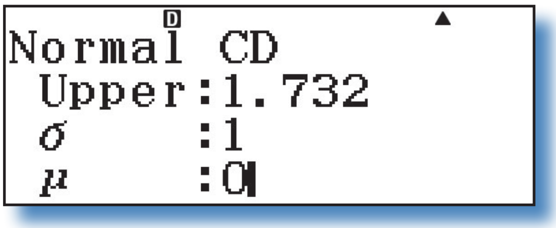


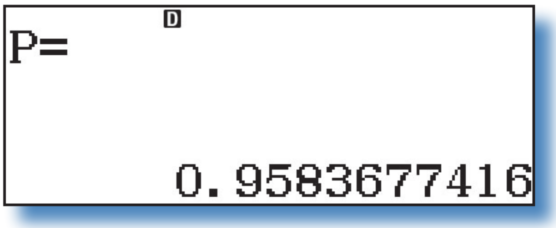


It means that $P(X \geq 92) = 1 - P(X < 92) = 1 - F(92) \approx 1 - 0.9584 = 0.0416$.

Extension:

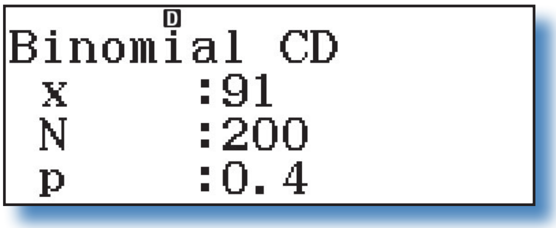
We can get the same result after standardization and using tables or the calculator with the standardized data.

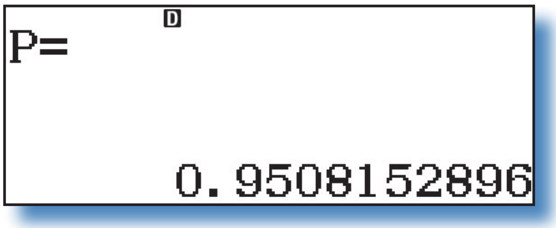




$$P(X \geq 92) = P(X \geq 92) = P\left(\frac{X - E}{D} \geq \frac{92 - 80}{\sqrt{48}}\right) =$$
$$= P\left(X_{std} \geq \sqrt{3}\right) = 1 - P\left(X_{std} < \sqrt{3}\right) \approx 1 - \Phi\left(\sqrt{3}\right) \approx 1 - 0.9584 = 0.0416$$

d) With the Casio calculator we can determine the exact value of the probability $P(X \geq 92)$. The distribution is binomial as we stated earlier and we can use Binomial Cumulative Distribution function of Casio calculator.





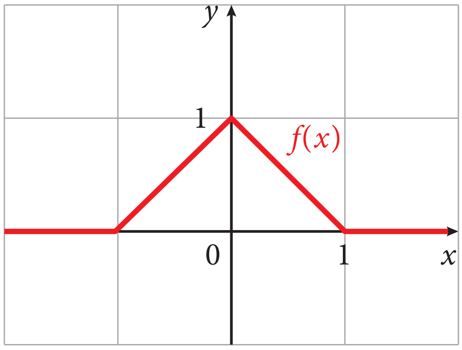
$P(X \geq 92) = 1 - P(X < 92) = 1 - F(91) \approx 1 - 0.9508 = 0.0492$

You can see that we got the best estimation using the Central limit theorem.

Problem 3.2.

Let the X_1, X_2, ... X_{100} random variables are independent and identically distributed (iid). Their probability density function (pdf) is

$$f(x) = \begin{cases} 1-|x|, & \text{if } x \in [-1;\ 1] \\ 0 & \text{otherwise} \end{cases}$$



Give an assumption for the probability $P(X_1 + X_2 + ... + X_{100} \geq 10)$ with the Chebyshev inequality!

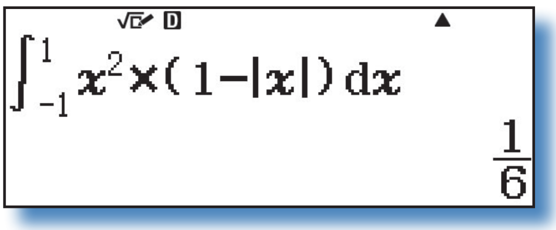
Solution of Problem 3.2.

We need to determine $E(X_i)$ the expected value of X_i.

Theoretically it is the integral $\int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-1}^{1} x(1-|x|)\, dx$ but now it is 0, because the $1-|x|$ function is symmetric for the y-axis (even function).

The standard deviation is $D^2(X) = E(X^2) - E^2(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \left(\int_{-\infty}^{\infty} x \cdot f(x)\, dx \right)^2$

If we use our integral function on the Casio calculator we get that $D^2(X) = \frac{1}{6}$.



Extension:

We can also get the integral after a short calculation.

$$D^2(X) = \int_{-1}^{1} x^2 \cdot (1-|x|)\, dx = \int_{0}^{1} 2x^2 \cdot (1-x)\, dx = \int_{0}^{1} 2x^2 - 2x^3\, dx = \left[\frac{2x^3}{3} - \frac{2x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{2}{4} = \frac{1}{6}$$

Let us turn back to the original problem.

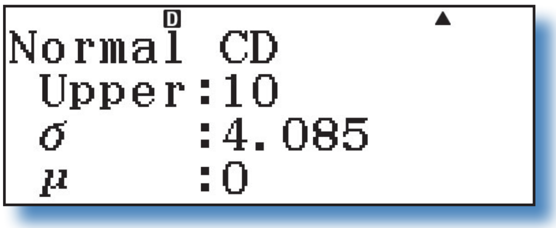
$$P(X_1 + X_2 + ... + X_{100} \geq 10) = \frac{1}{2} P(|X_1 + X_2 + ... + X_{100}| \geq 10) \leq \frac{1}{2} \cdot \frac{D^2(X_1 + X_2 + ... + X_{100})}{10^2} = \frac{1}{2} \cdot \frac{100 \cdot \frac{1}{6}}{10^2} = \frac{1}{12}$$

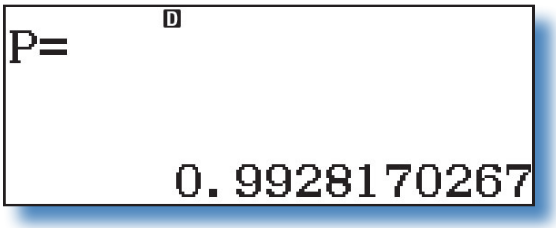
3. Markov and Chebyshev inequalities

We can use the Central limit theorem here again. Let us denote $Z = X_1 + X_2 + \ldots + X_{100}$.

We calculated earlier $\mu = E(Z) = 0$ and $\sigma^2 = D^2(Z) = \frac{100}{6} \approx 16.6667 \quad \Rightarrow \quad \sigma = D(Z) \approx 4.0825$.

Using these parameters and the fact $P(Z \geq 10) = 1 - P(Z < 10)$ with the help of menu 7 of the Casio calculator we get





$P(Z \geq 10) = 1 - P(Z < 10) \approx 1 - 0.9928 = 0.0072$

Extension:

If we insist on the standardization then after a short calculation we can get the same result.

$$P(Z \geq 10) = 1 - P(Z < 10) = 1 - P\left(Z_{\text{std}} < \frac{10}{\sqrt{\frac{100}{6}}} \right) = 1 - P\left(Z_{\text{std}} < \sqrt{6} \right) \approx 1 - 0.9928 = 0.0072$$

The estimation with the Chebyshev inequality gave the probability is $\frac{1}{12} \approx 0.0833$ and this is quite a magnitude worse then the estimation with the Central limit theorem.